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Mathematical modeling to traffic assignment problem and bridge location problems via fuzzy analysis

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Abstract : In this paper we introduce results of D. Han and H. K. Lo (2004) on traffic assignment problems and variational inequality problems. We discuss modeling of bridge location problems y applying the fuzzy theory.

1. Variational inequality problems

Let Ω be a nonempty closed convex subset in the n-dimensional linear space $\mathbf{R}^n$. Let $F: \Omega \to \mathbf{R}^n$ be a mapping and consider the following variational inequality problem :

$$(v - u^*)^T F(u^*) \geq 0 \quad \text{for } \forall v \in \Omega. \qquad \text{(VI)}$$

Here T is the transpose and $u^* \in \Omega$ is an optimal solution for (VI).
Denote the projection by

$$P_\Omega(x) = \{y^* \in \Omega : \| y^* - x \| = \min_{y \in \Omega} \| y - x \| \}.$$

Fig.1 shows that the following inequality of the projection:

$$(z - P_\Omega(z))^T (v - P_\Omega(z)) \leqq 0 \text{ for } z \in \mathbf{R}^n,\ v \in \Omega$$

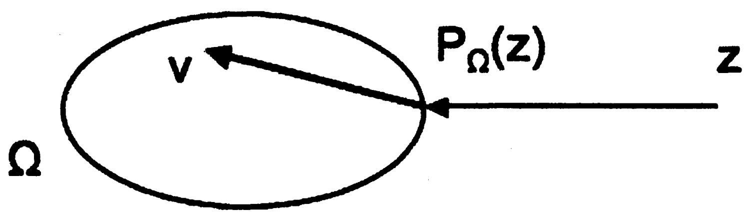


Fig.1. projection P_Ω

The equivalence conditions for optimal solution to (VI) holds as follows (see[1]):

u^* is optimal solution for $(v - u^*)^T F(u^*) \geq 0$ for $v \in \Omega$

$\Leftrightarrow$ Denoting $e(u,\beta) = u - P_{\Omega}(u - \beta F(u))$ for $\beta > 0$ it holds that $\| e(u^*,\beta) \| = 0$.

2. Solution algorithm

Ha-Lo[1] show the descent method as follows:

Theorem 1. Assume that F is a co-coercive mapping. Let parameters be

$0<\varepsilon<1$, $\mu>0$, $\{\beta_k\}: \beta_{min}>0$, $\beta_k \leq 4\mu(1-\varepsilon)$ and $0<\delta<2$.

Denote the residual function e and the iteration $\{u_k\}$ by

$$e(u,\beta_k) = u - P_{\Omega}(u - \beta_k F(u)) \text{ and } u_{k+1} = u_k - t_k e(u_k,\beta_k) \qquad \text{(A)}$$

with $t_k = \delta(1-(\beta_k/4\mu))$. Then the iteration $\{u_k\}$ is bounded and convergent to an optimal solution for (VI).□

Properties of co-coercive or monotone mappings play important roles to guarantee the optimal solutions for (VI)..

Theorem 2[1]. The following statements (a)-(d) hold.

(a) F is said to be monotone on Ω, if

$$(u-v)^T(F(u)-F(v)) \geqq 0$$

for $u,v \in \Omega$. Then (VI) has at least one optimal solution $\in \Omega$.

(b) F is said to be strictly monotone on Ω, if

$$(u-v)^T(F(u)-F(v)) > 0$$

for $u,v \in \Omega$, $u \neq v$. Then (VI) has the only optimal solution $\in \Omega$.

(c) F is said to be strongly monotone on Ω with modulus $\gamma > 0$, if

$$(u-v)^T(F(u)-F(v)) \geqq \gamma \| u-v \|^2$$

for $u,v \in \Omega$. Then (VI) has the only optimal solution in Ω.

(d) F is said to be co-coercive on Ω with modulus $\gamma > 0$, if

$$(u-v)^T(F(u)-F(v)) \geqq \gamma \| F(u)-F(v) \|^2$$

for $u,v \in \Omega$. Then (VI) has the only optimal solution in Ω.□

The following theorem shows that the majorant residual function means the linear convergence of (A) to the optimal solution for (VI).

Theorem 3[1]. Suppose that conditions of Theorem 2 hold. Furthermore, suppose that the residual function $e(u,\beta)$ with $\beta > 0$ a constant, provides an error bound for the optimal solution u^* to (VI) such that

$$\| u - u^* \| \leqq \xi \| e(u,\beta) \| \quad \text{for } u \in \Omega. \qquad \text{(C)}$$

Then the algorithm (A) converges linearly with

$$\| u_{k+1} - u^* \|^2 \leq [1-\{\delta(2-\delta)\varepsilon^2/(\xi^2\sigma)\}] \| u_k - u^* \|^2,$$

where $\sigma = 1$ (if $\beta < \beta_{min}$); β^2/β^2_{min} (otherwise). □

Remark. Sufficient conditions for (C) to hold have been proposed in [2] and references therein.

3. Traffic assignment problems

While the descent algorithm is applicable to variational inequality problem (VI) with co-coercive mappings, this section specializes it for traffic assignment problems. Consider a strongly connected transportation network, i.e., containing at least one directed route from every node to every other node.

Denote the following notations.

$G(N,A)$: strongly connected transportation network

N : set of nodes

$A = \{a\}$: set of links

$RS = \{ rs \}$: set of origin-destination (O-D) pairs

$P^{rs} = \{p\}$: set of routes connecting rs

$H = \{h\}$: set of feasible route flow vector

Defining $M = (\delta_{ap})$ as the link-route incidence matrix such that

$\delta_{ap} = 1$ (if a route p contains link a) ; $\delta_{ap} = 0$ (otherwise).

$F = MH = \{ f = Mh : h \in H\}$

$C_a(f)$: link cost function for $a \in A$ and $f = Mh$

$C(h)$: route cost function of h such that

$C(h) = (C_1(f), C_2(h), \ldots, C_r(h))^T$, where $r = |M|$ is the number of links of A.

The equilibrium traffic assignment problem can be formulated by the following variational inequality problem:

$$\text{find } f^* \in F \text{ such that } (f - f^*)^T C(f^*) \geqq 0 \quad \text{for } f \in F.$$

Or equivalently, find $h^* \in H$ ($f^* = Mh^*$) such that

$$(h - h^*)^T M^T C(Ah^*) \geqq 0 \text{ for } h \in H.$$

The following lemma shows that the mapping M^TCM is so as C is co-coercive.

Lemma[1]. If C is a co-coercive mapping with modulus μ, i.e.,

$$(u-v)^T C(u-v) \geq \mu \,||C(u-v)||^2$$

for $u, v \in \mathbf{R}^n$, then it follows that

$$(u-v)^T M^T CM(u-v) \geq (\mu/||M||)^2 ||M^T CM(u-v)||^2$$

for $u, v \in \mathbf{R}^n$.

4. Example of non-fuzzy traffic assignment problems

Ha-Lo [1] discussed a transportation network with two origins(1 and 4), two destinations(2 and 3), additionally 9 nodes and 19 links. See Fig. 2.

There are two types of the link cost functions as follows:

1) affine type with $C(f) = Pf + b$ with constant matrix P and vector b;

2) nonlinear type with $C(f) = (c_i(f))$, where $c_i(f) = a_i\,(1 + b(f_i / V_i)^d)$

$f_i = 0,1$ (f: route vector), a_i : free travel cost of link i, b : coefficient

V_i: traffic capacity of i, d : magnitude of the congestion effect (d=3,4).

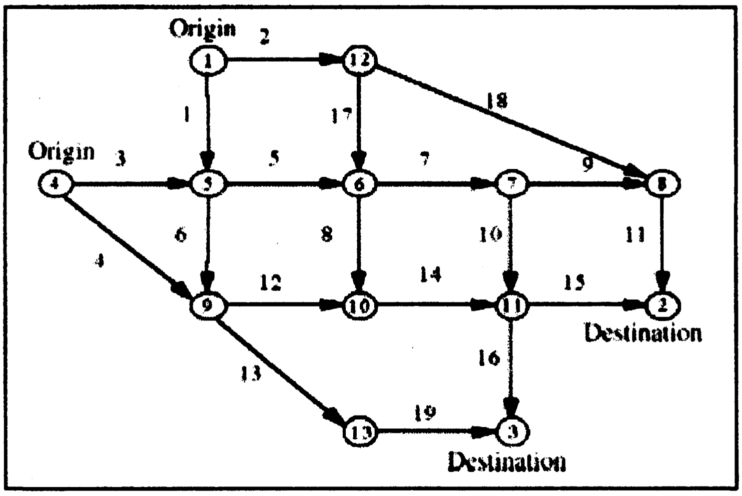
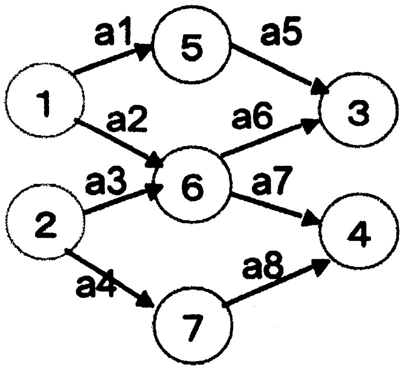


Fig. 2 Transportation network with four origins and destinations.

In the other example we illustrate a transportation network with two origins (1 and 2), two destinations (3 and 4), additionally 3 nodes and 8 links. See Fig. 3. There are six routes from one origin to one destination: route 1 with links a1 and a5 starts from origin1 to ends at destination 3, etc as the below table.



Route p	Link a
1	a1, a5
2	a2, a6
3	a2, a7
4	a3, a6
5	a3, a7
6	a4, a8

Fig. 3

Then the set of links is

$$A = \{a1, a2, a3, a4, a5, a6, a7, a8\}$$

and the number of the links is $|A| = 8$. In this case we have the following variational inequality problem

$$(f - f^*)^T C(f^*) \geq 0$$

for $f \in F$. Here F is the set of feasible route vectors and f = Mh, M is the incident matrix as follows:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

f

The vector $f = (1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ )^T \in \mathbf{R}^8$ with $f = Me_1$ and $e_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^T$. $C(f) = (c_1(f), c_2(f), \ldots, c_8(f))^T$ is the link cost function.

5.Bridge location problem of fuzzy case

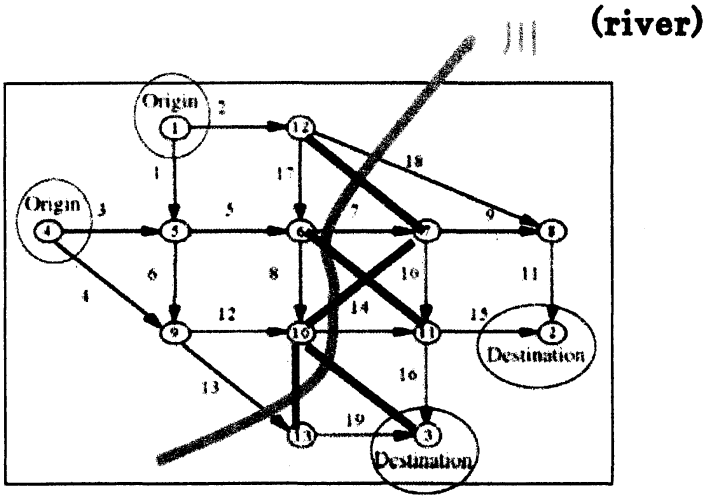
In this paper we discussed the following two points.

1) The traffic assignment problem with the link cost function C(f) of link f passing through from one origin to one destination is reduced to the variational inequality problem (VI):

$$(f - f^*)^T C(f^*) \geqq 0.$$

The mapping C with monotone or co-coercive conditions gives linear convergence to an optimal solution of (VI).

2) The above modeling in traffic assignment problems is fairly useful in modeling to bridge location problems(see Fig. 4). Where are the best location of bridges over the river under an objective function attains the optimal value? In this case we have some questions which is the objective function to the bridge location problems where the traffic volume of streets (links between two nodes) has the fuzzy number, etc. It is considered that the fuzzy numbers are L-fuzzy numbers, i.e, symmetry types. It is expected that jam function is considered as the objective function. The Jam situation of streets in Japan is considered from three viewpoints: traffic velocity(average), traffic length and time of keeping the situation. On the other hand a new denition of traffic jam is considered with traffic volume = velocity * density (see [3] and refences therein).



橋の位置を示す (Location of bridges)

Fig. 4 Bridge location problem

References

[1] D. Han and H. K. Lo: Solving non-additive traffic assignment problems: A descent method for co-c-ercive variational inequalities, European J. OR, 159, 2004, pp.524-544.

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